

Closing Wed: HW\_8A,8B (8.1)  
 Closing *next* Wed: HW\_9A,9B (8.3,9.1)  
 No lecture this Wednesday!

**Chapter 8: More Applications.**

**3.1 Arc Length (we already discussed)**

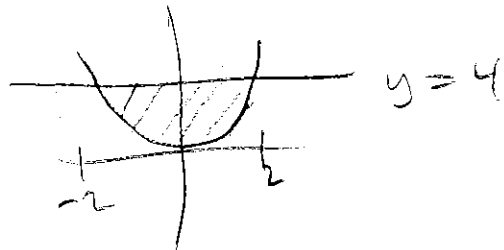
**3.3 Center of Mass**

For your own interest read (not on test):

- 3.2: Surface area
- 3.3: Hydrostatic (water) pressure & force
- 3.4: Economics and biology apps
- 3.5: Probability apps (bell curve)

**3.3 Center of Mass**

Goal: Given a thin plate (a *lamina*) where the mass is uniformly distributed, we find the center of mass (*centroid*).



If  $y = f(x) = \text{"top"}$ ,  $y = g(x) = \text{"bottom"}$ , then the center of mass (centroid) is

$$\bar{x} = \frac{1}{\text{Area}} \int_a^b x(f(x) - g(x)) dx$$

$$\bar{y} = \frac{1}{\text{Area}} \int_a^b \frac{1}{2} [(f(x))^2 - (g(x))^2] dx$$

Example: Find the centroid of the region bounded by  $y = x^2$  and  $y = 4$ .

$$\begin{aligned} \text{AREA} &= \int_{-2}^2 4 - x^2 dx = 4x - \frac{1}{3}x^3 \Big|_{-2}^2 \\ &= (8 - \frac{8}{3}) - (-8 + \frac{8}{3}) \\ &= 16 - \frac{16}{3} = \frac{32}{3} \end{aligned}$$

$$\bar{x} = \frac{1}{32/3} \int_{-2}^2 x(4 - x^2) dx = 0$$

$$\begin{aligned} \bar{y} &= \frac{1}{32/3} \int_{-2}^2 \frac{1}{2} [(4)^2 - (x^2)^2] dx \\ &= \frac{3}{32} \left( \frac{88}{3} \right) = \frac{88}{32} = \frac{11}{4} = 2.75 \end{aligned}$$

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Derivation (don't need to write)

if you are given  **$n$  points**

$(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$  with masses

$m_1, m_2, \dots, m_n$

then

$$M = \text{total mass} = \sum_{i=1}^n m_i$$

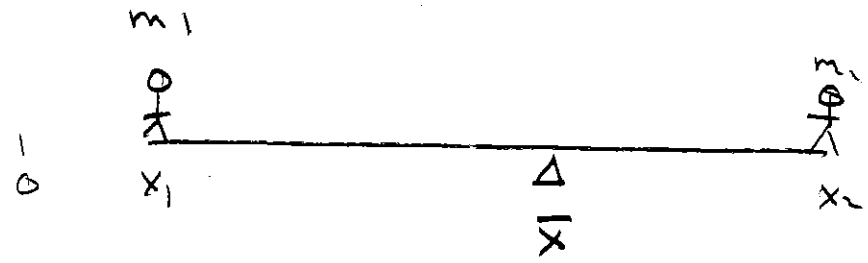
$$M_y = \text{moment about } y \text{ axis} = \sum_{i=1}^n m_i x_i$$

$$M_x = \text{moment about } x \text{ axis} = \sum_{i=1}^n m_i y_i$$

$$\bar{x} = \frac{m_1 x_1 + \dots + m_n x_n}{m_1 + \dots + m_n} = \frac{M_y}{M}$$

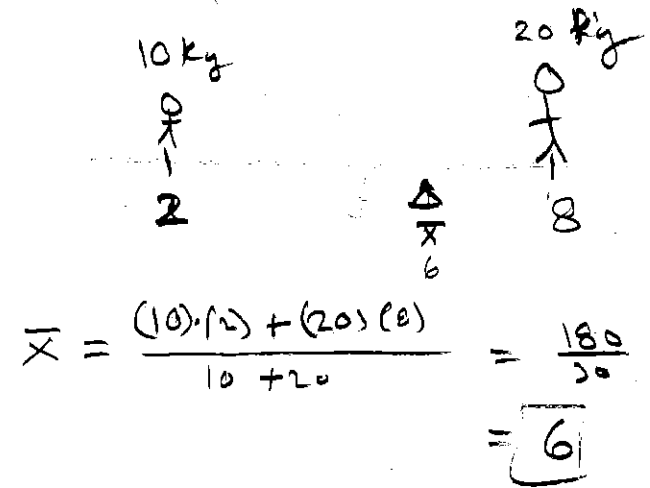
$$\bar{y} = \frac{m_1 y_1 + \dots + m_n y_n}{m_1 + \dots + m_n} = \frac{M_x}{M}$$

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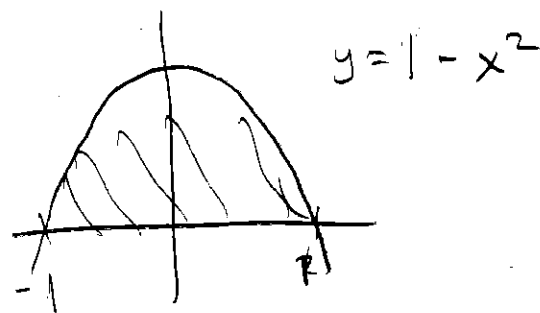
$$m_1 (\bar{x} - x_1) = m_2 (x_2 - \bar{x})$$
$$m_1 \bar{x} - m_1 x_1 = m_2 x_2 - m_2 \bar{x}$$
$$\Rightarrow (m_1 + m_2) \bar{x} = m_1 x_1 + m_2 x_2$$
$$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

Ex)



Example:

Find the center of mass (centroid) of a thin plate with uniform density  $\rho = 2 \text{ kg/m}^2$  that looks like the region bounded by  $y = 4 - x^2$  and the x-axis.



$$\begin{aligned} \text{AREA} &= \int_{-1}^1 (4 - x^2) dx \\ &= \left[ 4x - \frac{1}{3}x^3 \right]_{-1}^1 = (4 - \frac{1}{3}) - (-4 + \frac{1}{3}) = 8 - \frac{2}{3} = \frac{22}{3} \end{aligned}$$

$$\text{TOTAL MASS} = 2 \frac{\text{kg}}{\text{m}^2} \cdot \frac{22}{3} \text{ m}^2 = \frac{44}{3} \text{ kg} = M$$

$$\int_{-1}^1 x(4 - x^2) dx = \int_{-1}^1 (4x - x^3) dx = 0 \Rightarrow M_y = 2(0) = 0$$

$$\int_{-1}^1 \frac{1}{2} (4 - x^2)^2 dx = \frac{1}{2} \int_{-1}^1 (16 - 16x^2 + 4x^4) dx$$

$$= \frac{1}{2} \left[ 16x - \frac{16}{3}x^3 + \frac{4}{5}x^5 \right]_{-1}^1$$

$$= \frac{1}{2} \left[ \left( 16 - \frac{16}{3} + \frac{4}{5} \right) - \left( -16 + \frac{16}{3} - \frac{4}{5} \right) \right]$$

$$= \frac{1}{2} \left[ \frac{30 - 16 + 4}{3} + \frac{16 - 16 + 4}{5} \right] = \frac{1}{2} \left[ \frac{18}{3} + \frac{4}{5} \right]$$

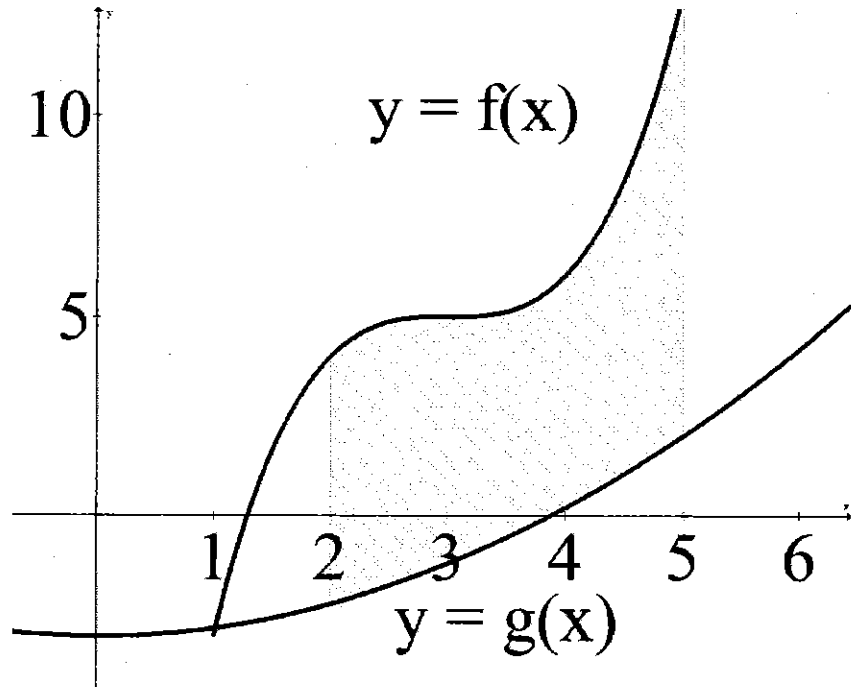
$$= \frac{1}{2} \cdot \frac{16}{15} = \frac{8}{15}$$

$$M_x = 2 \cdot \frac{8}{15} = \frac{16}{15}$$

$$\bar{x} = \frac{M_y}{M} = \frac{0}{\frac{44}{3}} = 0$$

$$\bar{y} = \frac{M_x}{M} = \frac{\frac{16}{15}}{\frac{44}{3}} = \frac{2}{11}$$

If the region is bounded between two curves,



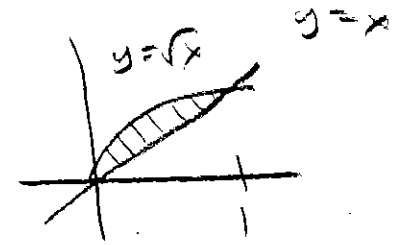
what changes in derivation?

$$\bar{x} = \frac{p \int_a^b x(f(x) - g(x)) dx}{p \int_a^b f(x) - g(x) dx}$$

$$\bar{y} = \frac{p \int_a^b \frac{1}{2} [(f(x))^2 - (g(x))^2] dx}{p \int_a^b f(x) - g(x) dx}$$

*Example:*

Find the center of mass (centroid) of a thin plate with uniform density  $\rho = 3 \text{ kg/m}^2$  that looks like the region bounded by  $y = x$  and  $y = \sqrt{x}$ .



$$\text{AREA} = \int_0^1 \sqrt{x} - x dx = \frac{1}{6} \Rightarrow m = \frac{3}{6} = \frac{1}{2}$$

$$\int_0^1 x(\sqrt{x} - x) dx = \frac{1}{15} \Rightarrow m_y = \frac{3}{15} = \frac{1}{5}$$

$$\int_0^1 \frac{1}{2} [(\sqrt{x})^2 - (x)^2] dx = \frac{1}{12} \Rightarrow m_x = \frac{2}{12} = \frac{1}{6}$$

$$\bar{x} = \frac{m_y}{m} = \frac{1/5}{1/2} = \frac{2}{5}$$

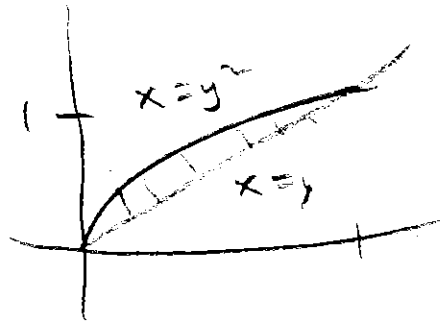
$$\bar{y} = \frac{m_x}{m} = \frac{1/6}{1/2} = \frac{1}{3}$$

What if we want to do everything in terms of  $y$  instead of  $x$ , then swap "x" and "y" **everywhere** in these formulas.

$f(y)$  = right bound,  $g(y)$  = left bound, then the center of mass is given by

$$\bar{y} = \frac{p \int_a^b y(f(y) - g(y)) dy}{p \int_a^b f(y) - g(y) dy}$$

$$\bar{x} = \frac{p \int_a^b \frac{1}{2} [(f(y))^2 - (g(y))^2] dy}{p \int_a^b f(y) - g(y) dy}$$



**Example:**

Find the center of mass (centroid) of a thin plate with uniform density  $\rho = 3 \text{ kg/m}^2$  that looks like the region bounded by  $y = x$  and  $y^2 = x$ .

(same problem from previous page, but now do it in terms of  $y$ .)

$$\text{Area} = \int_0^1 y - y^2 dy = \left. \frac{1}{2}y^2 - \frac{1}{3}y^3 \right|_0^1$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \checkmark$$

$$\int_0^1 y(y - y^2) dy = \frac{1}{12}$$

$$\int_0^1 \frac{1}{2} [(y)^2 - (y^2)^2] dy = \frac{1}{15}$$

$$\bar{y} = \frac{1/12}{1/6} = \frac{1}{2}$$

$$\bar{x} = \frac{1/15}{1/6} = \frac{2}{5}$$

## Just for your own interest:

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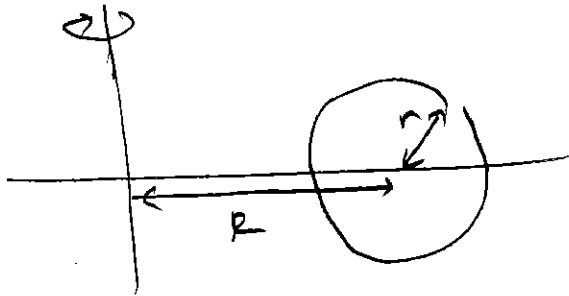
### Theorem of Pappus

The volume of a solid of revolution is equal to the product of the area of the region,  $A$ , and the distance traveled by the center of mass of the region around the axis of rotation,  $d$ . (Note:  $d = 2\pi\bar{x}$ )

Thus,  $\text{Volume} = (\text{Area})2\pi\bar{x}$

### Quick Application:

Find the volume of the torus.



↑  
CENTER OF  
MASS OF  
CIRCLE =  $(R, 0)$

### Proof

Using the shell method, we get:

$$\begin{aligned}\text{Volume} &= \int_a^b 2\pi x(f(x) - g(x))dx \\ &= 2\pi \int_a^b x(f(x) - g(x))dx\end{aligned}$$

From today:

$$\bar{x} = \frac{\int_a^b x(f(x) - g(x))dx}{\text{Area}}, \quad \text{so}$$

$$2\pi \int_a^b x(f(x) - g(x))dx = 2\pi\bar{x}(\text{Area}).$$

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$$\text{AREA OF CIRCLE} = \pi r^2$$

DIST. TRAVELLED

$$\text{BY } (r, 0) = 2\pi R$$

IN  
ONE REVOLUTION

$$(\text{AREA}) \cdot \text{DIST. TRAVELLED} = \pi r^2 \cdot 2\pi R = 2\pi^2 r^2 R$$